LANDSCAPE METRICS SELECTION BASED ON THE MATHEMATICAL MODELS OF LANDSCAPE PATTERNS

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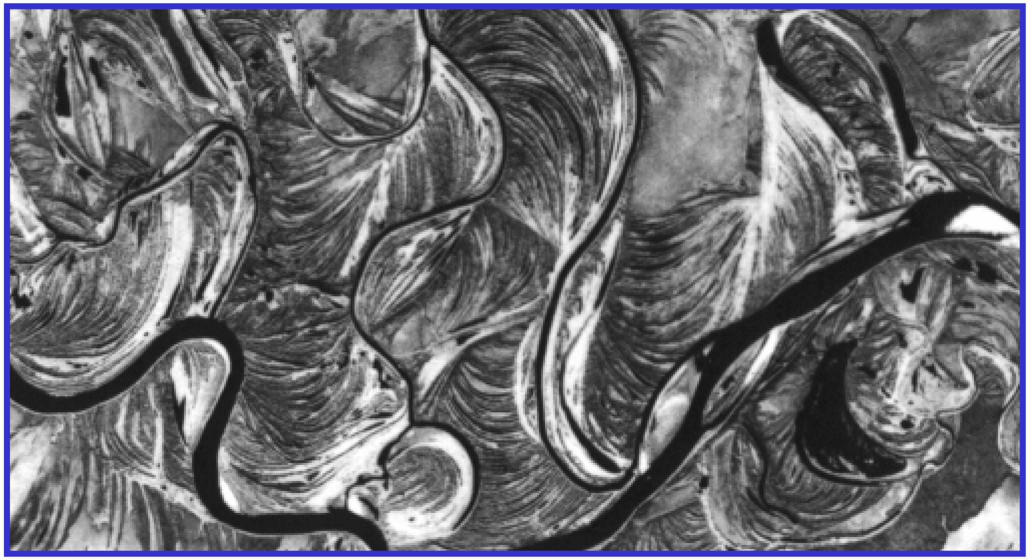
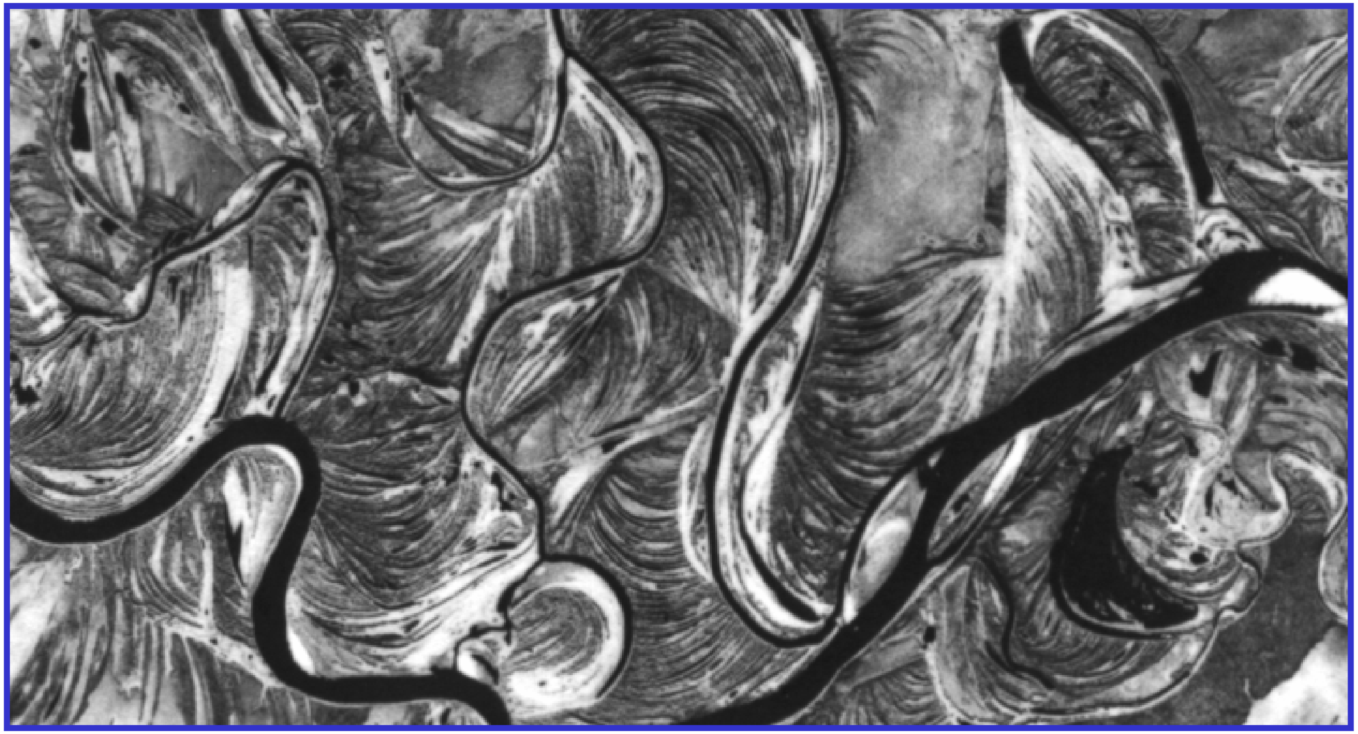


Many researchers including V.A. Nikolaev, L.I. Ivashutina, Yu.G. Simonov, V.M. Fridland, B.V. Vinogradov, N.V. Fadeeva, K.H.Ritters, R.V. O'Neill, K. McGarigal, B.J. Marks, and others design different quantitative parameters of landscape mosaics.



The landscape metric - quantitative parameter of landscape mosaics formed by natural units on the Earth surface.

Landscape metrics can be used for different tasks:

- **Automatic image fragmentation,**
- **General characteristics of spatial landscape structure,**
- **Obtaining data on geological conditions,**
- **Analysis for sustainable development (landscape measurement).**
- **......**

A great number of similar parameters are used now in publications.

Ivashutina LI, Nikolaev VA (1971) Landscape pattern contrast and some aspects of its study. In: Vestnik MSU, geogr. # 5, pp 77–81 (in Russian)

Victorov AS (1986) Risunok landshafta (Landscape pattern), Publ. House "Mysl'", Moscow, p 179.

Riitters KH, O'Neill RV, Hunsaker CT, Wickham JD, Yankee DH, Timmins SP, Jones KB, Jackson BL (1995) A vector analysis of landscape pattern and structure metrics. Landscape Ecol 10:23 – 39

Leitao A.B. et al. (2006) Measuring landscapes: a planner's handbook. Island press, p.245

Moser B., Jaeger J. A.G., Tappeiner U., Tasser E., Eiselt B..(2006) Modification of the effective mesh size for measuring landscape fragmentation to solve the boundary problem. In: Landscape Ecology 3/22.

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A great number of similar parameters are used now in software packages of landscape mosaic analysis .

McGarigal K, Cushman SA, Neel MC, Ene. E. (2002) FRAGSTATS: Spatial Pattern Analysis Program for Categorical Maps. Computer software program produced by the authors at the University of Massachusetts, Amherst, MA: University of Massachusetts. www.umass.edu/landeco/research/fragstats/fragstats.htm

Pshenitchnikov A.E. (2003) Thematic morphometry – general directions and used parameters. In: Vestnik MSU, geogr. 5: 42–46 (in Russian)

Table . Quantitative parameters of landscape mosaics (fragment)

Ratio of contour types .	$S_1 : S_2 : ... : S_m$	1907	Н.А. Димо Б.А. Келлер
"Fraction index ", "Areal heterogeneity coefficient" "Fraction index"	$n_o = \frac{n}{S}$	1964 1966 1972	М.А. Глазовская С.В. Викторов В.М. Фридланд
Entropic measure of landscape pattern diversity	$k = n_0 \frac{S - q_{\max}}{S}$ $H = -\sum_{i=1}^{m} \frac{S_i}{S} \log_2 \frac{S_i}{S}$	1968 1970 1971	Б.Л. Гуревич К,И, Геренчук А,Г, Топчиев Ю,Г, Симонов
"Landscape heterogeneity coefficent"	$k_н = \frac{\sum_{j=1}^{m^*-1} \sum_{i=j+1}^{m^*} \frac{m^* S_i}{S} \frac{m^* S_j}{S}}{\frac{m^*}{2}}$	1969	Л.И. Ивашутина В.А. Николаев
Complexity index	$R = \frac{1}{S} \left(\frac{m}{n} \right)^{\alpha} HK_2$	1982	Ю.М.Россааре
"Degree of patch size differentiation"	$k = \frac{\sum_{j=1}^{n} \left\| q_j - q_0 \right\|}{n q_0}$	1969	Ostrovskiy Jankovskiy
Roundness index	$k_m = \frac{4\pi \quad q}{P^2}$	1953	V.C. Miller
Shape parameter	$k = \frac{q}{d^2}$	1932	R.E. Horton
"Elliptic parameter"	$k = \frac{\pi d^2}{4q}$	1965	D.R .Stoddart
Anisometric parameter	$k = \frac{1}{d} \sqrt{\frac{q}{\pi}}$	1956	S.A. Schum
Tortuosity coefficient	$k = \frac{P}{q}$	1852	C. Ritter
…………			

Victorov AS (1998) Mathematical morphology of landscape (Matematicheskaya morfologiya landshafta). Publ. House «Tratek», Moscow, p 192

Quantitative parameters of landscape mosaics

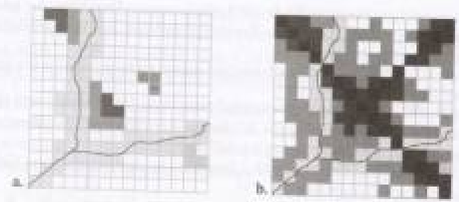

Figure 3.7. A comparison of a landscape (a) with low patch numbers and patch density with a landscape (b) possessing higher patch numbers and patch density. The two landscapes have a mixture of land cover types (LCTs): roads (black), urban (dark gray), agriculture (medium gray), riparian (light gray), forest (white), meadow (cross hatch), and wetland (stipple). Landscape (a) has nine patches (PN) and a PD of 44 patches per 100 hectares, while landscape (b) has 84 patches (PN) and a PD of 415 patches per 100 hectares.

smaller patches. In landscape (a) these same LCTs are in large contiguous patches, often more simply shaped.

3.3.2. *Metric Equation and Calculation*

PN is simply a count of all the patches within a class or across the entire landscape. PD is the normalized form of PN—achieved by dividing PN by total landscape area.

$$PN = \sum_{i=1}^{n} P_i$$

P_i = *patch of type* i

$$PD = \frac{PN}{A} \times (10{,}000)^{m^2}/_{ha} \times 100$$

A = total landscape area in m^2

Units: None
Range of values: 1, up to the total number of grid cells in the landscape dataset, where each grid cell would be a separate patch.

The following calculation is an example for landscape (a) in figure 3.7. (Note: there are 15 × 15 grid cells, 30 m by 30 m each.)

$$PD = \frac{9}{(202{,}500m^2)} \times (10{,}000)^{m^2}/_{ha} \times 100$$

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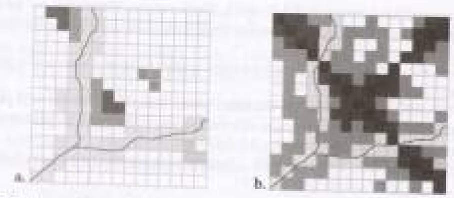
Quantitative parameters of landscape mosaics



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Quantitative parameters of landscape mosaics

dependent upon agricultural ecosystems (including farmers) will likely function better in landscape (b), as these patches are more proximal to one another than they are in landscape (a).

3.10.2. *Metric Equation and Calculation*

PROX is calculated for each patch by summing the area of each neighboring patch of the same LCT that lies within a specified neighborhood distance from the focal patch, after weighting the area of each neighboring patch by its distance from the focal patch.

$$PROX = \sum_{s=1}^{n} \frac{a_{ijs}}{b_{ijs}^2}$$

Where a_{ijs} = the area of the *s*th? patch within the specified search radius of patch *ij* and b_{ijs} = the distance from patch *ij* to the *s*th? neighboring patch of the same type, based upon edge-to-edge distance.

Units = dimensionless
Range of values > 0, without limit

Figure 3.38 below illustrates the calculation of PROX for a focal patch within a landscape (represented by a fifty-cell square grid).

$$PROX = \sum_{s=1}^{n} \frac{a_{ijs}}{b_{ijs}^2} = \frac{9}{3^2} + \frac{46}{5^2} + \frac{50}{5.83^2} + \frac{5}{10^2} + \frac{1}{5.83^2} + \frac{1}{2^2} + \frac{107}{3^2} = 16.529$$

This example illustrates the application of PROX to raster landscape data. It is also possible to apply PROX to polygonal landscape data with similar computational results, however it is easier to analyze raster data, due to the inherent spatial indexing (row and column) of raster datasets.

PROX results are dimensionless and can range upwards from zero with no limit. Larger values of PROX indicate that neighboring patches of the same LCT are larger and closer together, i.e., that patches are less isolated. Small values indicate that patches are further apart and may be smaller in area, i.e., that patches are more isolated. If a patch is the only one of its type within a landscape, the value for PROX is zero because it does not have any other patches of the same type that are proximate to it.

PROX is calculated at the patch level and can be summarized at the

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Quantitative parameters of landscape mosaics

$$GYRATE = \sum_{r=1}^{z}\left(\frac{h_{ijr}}{z}\right)$$

h_{ijr} = distance (m) between cell ijr (located within patch ij) and the centroid of patch ij (the average location), based on cell-center-to-cell-center distance.

z = number of cells in patch ij.

Units: distance units of the dataset (typically meters)
Range of values: ≥ 0, without limit.

GYRATE = 0 when the patch consists of a single cell and increases without limit as the patch increases in extent. GYRATE achieves its maximum value when the patch comprises the entire landscape. GYRATE is calculated for individual patches and can be summarized at the class and landscape levels. The most common summary is the Area-Weighted Mean Radius of Gyration (GYRATE_AM).

GYRATE_AM (area-weighted mean) at the class level equals the sum, across all patches of the corresponding patch type, of the radius of gyration (GYRATE) multiplied by the proportional abundance of the patch (i.e., patch area divided by the sum of all patch areas of the corresponding patch type). Units and range are the same as for GYRATE.

Class-level calculation of GYRATE_AM:

$$GYRATE\_AM = \sum_{j=1}^{n}\left[\sum_{r=1}^{z}\left(\frac{h_{ijr}}{z}\right)\left(\frac{a_{ij}}{\sum_{j=1}^{n} a_{ij}}\right)\right]$$

GYRATE_AM provides the analyst with a measurement of connectivity known as *correlation length*. Correlation length is the average distance one might traverse across the map while remaining within a patch from a random starting point and moving in a random direction. In other words, correlation length is the expected traversability of the map (Keitt et al. 1997). Large values of GYRATE_AM indicate more connected (less subdivided) landscapes.

Landscape-level calculation of GYRATE_AM (correction length):

$$GYRATE\_AM = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sum_{r=1}^{z}\left(\frac{h_{ijr}}{z}\right)\left(\frac{a_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}}\right)\right]$$

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Quantitative parameters of landscape mosaics

movement of organisms dependent upon forest cover (dark gray) would be expected to be fairly uninhibited. In addition, the role of forest cover in the types and rate of nutrient cycling would likely be significant. Landscape (b), on the other hand, has only 24% of the maximum aggregation. In other words, landscape (a) is 3.5 times more clumped than landscape (b). Landscape (b), with a lower contagion value, is spatially more diverse, primarily due to the spatial arrangement of its LCTs. Note that both landscapes have the same complement of LCTs. The LCTs in landscape (b) occur in many smaller patches than in landscape (a). There is also a greater degree of inter-digitation between contrasting LCTs in landscape (b). With smaller patches, particularly forest patches, it is unlikely that landscape (b) could support the same-sized populations of forest-dependent organisms as landscape (a). More complex edges (increased inter-digitation) also mean that the number of shared edges between contrasting LCTs is increased. Edges are often responsible for increased predation, invasion of exotic plant species and, in many cases, serve as barriers to movement for both plants and animals. In sum, landscape (b) would be expected to have fewer forest interior species (plant and animal), smaller populations of these species, and groups of these organisms (i.e., local population) would be relatively isolated from one another. In addition, forest and other native cover types would play a smaller role in nutrient cycling and storage. Given the important role of LCT pattern in these characteristics, and since both landscapes have the same complement of LCTs and comparable areas of forest and residential land cover, it is important to find a means of quantifying the spatial pattern differences between the landscapes. Contagion provides an objective means for quantifying this comparison.

3.7.2. *Metric Equation and Calculation*

Contagion describes the diversity of shared edges between different land cover types (LCTs) in a rasterized, or gridded dataset. At the landscape level, contagion sums, for all LCTs, the product of two distinct proportions for each LCT (McGarigal and Marks 1995). The first proportion is that of the landscape occupied by a particular cover type i represented by the term P_i (this is equivalent to class area proportion; see section 3.2). The second is the proportion of LCT adjacencies that involve any two LCTs, including like and different LCTs. For example, LCT i and LCT k are represented by the term P_{ik}. The equation is shown below. P_{ik} can be easily computed from an adjacency matrix (see Figure 3.26).

$$CONTAG = \left(1 + \frac{\sum_{i=1}^{m}\sum_{k=1}^{n}\left[(P_i)(P_{ik})\right]\cdot\ln\left[(P_i)(P_{ik})\right]}{C\max}\right)\times 100$$

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Quantitative parameters of landscape mosaics

Table 3.6. Amount of shared edge between LCTs in Figure 3.30.

Dark Gray and White edge: (80) x (0.90) =	72
Dark Gray and Light Gray edge: (150) x (0.50) =	75
Dark Gray and Medium Gray edge: (40) x (0.20) =	8
TOTAL	155

The numerator portion of the equation is calculated as shown in table 3.6. Next we divide the 155 by the total perimeter length (280) and multiply the quotient by 100 to produce the edge contrast percentage value:

$$155 \div 280 = 0.554$$
$$0.554 \times 100 = 55.4 \text{ percent}$$

Contrast can range between zero and 100%. If the dark gray patch had possessed maximum contrast edges around its entire perimeter (every segment having an edge contrast of one), then the value for ECON would be 100%. Conversely, if the dark gray patch had only a minimal edge contrast for all of its perimeter segments, then the value for ECON would be very close to zero percent.

Edge contrast can be measured at the patch, class, or landscape level. The calculation for an individual patch is described above. At the class and landscape levels, ECON can be calculated as the Mean Edge Contrast Index (ECON_MN) or MECI, Area-weighted Mean Edge Contrast Index (ECON_AM), and Total Edge Contrast Index (TECI). ECON-MN is calculated as the mean of the individual patch ECON values. At the class level, the result is the average edge contrast index for all the patches of a particular LCT within the landscape.

Class-level calculation of mean edge contrast index (ECON_MN):

$$ECON\_MN = \frac{\sum_{j=1}^{n}\left[\frac{\sum_{k=1}^{m'}(p_{ijk} \cdot d_{ik})}{p_{ij}}\right]}{n_i}(100)$$

ECON_MN (mean) at the class level equals the sum of all the patch perimeter segment lengths (m) involving the corresponding patch type multiplied by their corresponding contrast weights, divided by the total patch perimeter, summed across all patches of the corresponding type,

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Quantitative parameters of landscape mosaics

divided by the number of patches of the same type, multiplied by 100 (McGarigal and Marks 1995). Units and range are the same as for ECON.

Class-level calculation of area-weighted mean edge contrast (ECON_AM):

$$ECON\_AM = \sum_{j=1}^{n}\left(\left[\frac{\sum_{k=1}^{m}(p_{ijk}\cdot d_{ik})}{p_{ij}}\right]\left[\frac{a_{ij}}{\sum_{j=1}^{n}a_{ij}}\right]\right)(100)$$

ECON_AM (area-weighted mean) at the class level equals the sum of the segment lengths (m) of the perimeter of each patch multiplied by its corresponding contrast weights, divided by total patch perimeter, multiplied by patch area divided by the sum of patch areas, summed across all patches of the corresponding patch type with the product multiplied by 100. Units and range are the same as for ECON.

With this measurement, large patches are weighted more heavily than small patches in recognition of the fact that large patches often play a dominant role in how a landscape functions. This metric can be particularly useful in situations where one or more large patches of native ecosystem may be dominant within the landscape. Any edge effects associated with these patches may therefore have more of an impact upon the ecology of the landscape than edge effects associated with very small patches (small patches having less habitat value for fewer species).

An alternative to the patch-based metrics (ECON, ECON_MN, and ECON_AM) is TECI. This metric enables the investigator to examine all the edges of a particular LCT (class level), or over the entire landscape (landscape level), based solely upon the characteristics of each edge segment, independent of its patch affiliation.

Class-level calculation of total edge contrast index (TECI):

$$TECI = \frac{\sum_{k=1}^{m}(e_{ik}\cdot d_{ik})}{\sum_{k=1}^{m}e_{ik}}(100)$$

e_{ik} = total length (m) of edge in landscape between patch types (classes) i and k; includes landscape boundary segments involving patch type i.

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In fact a number of possible metrics is infinite, so their number can be increased.

Optimization problem of metrics selection

The problem includes the following aspects:

- **What metrics among their endless number should be chosen for a given task?**
- **What metrics taken together give us additional data and what do not?**
- **How can we minimize a number of metrics for a certain task without loosing information?**

The importance of the tasks: A little example

a

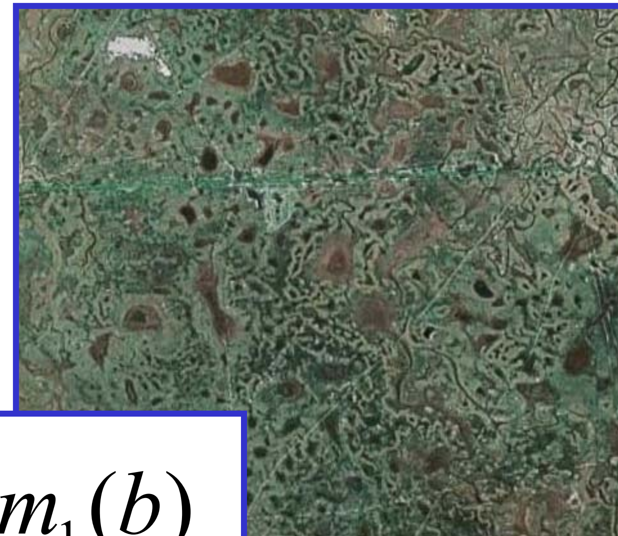
$$m_1(a) < m_1(b)$$

b

$$m_2(a) \ ? \ m_2(b)$$

The importance of the tasks: A little example

a



b

$$m_1(a) < m_1(b)$$

$$m_2(a) < m_2(b)$$

$$m_2 = F(m_1)$$

The false result

The aim of the report

The aim of the report is to show that decision of the assigned tasks can be based on the mathematical models of spatial landscape patterns.

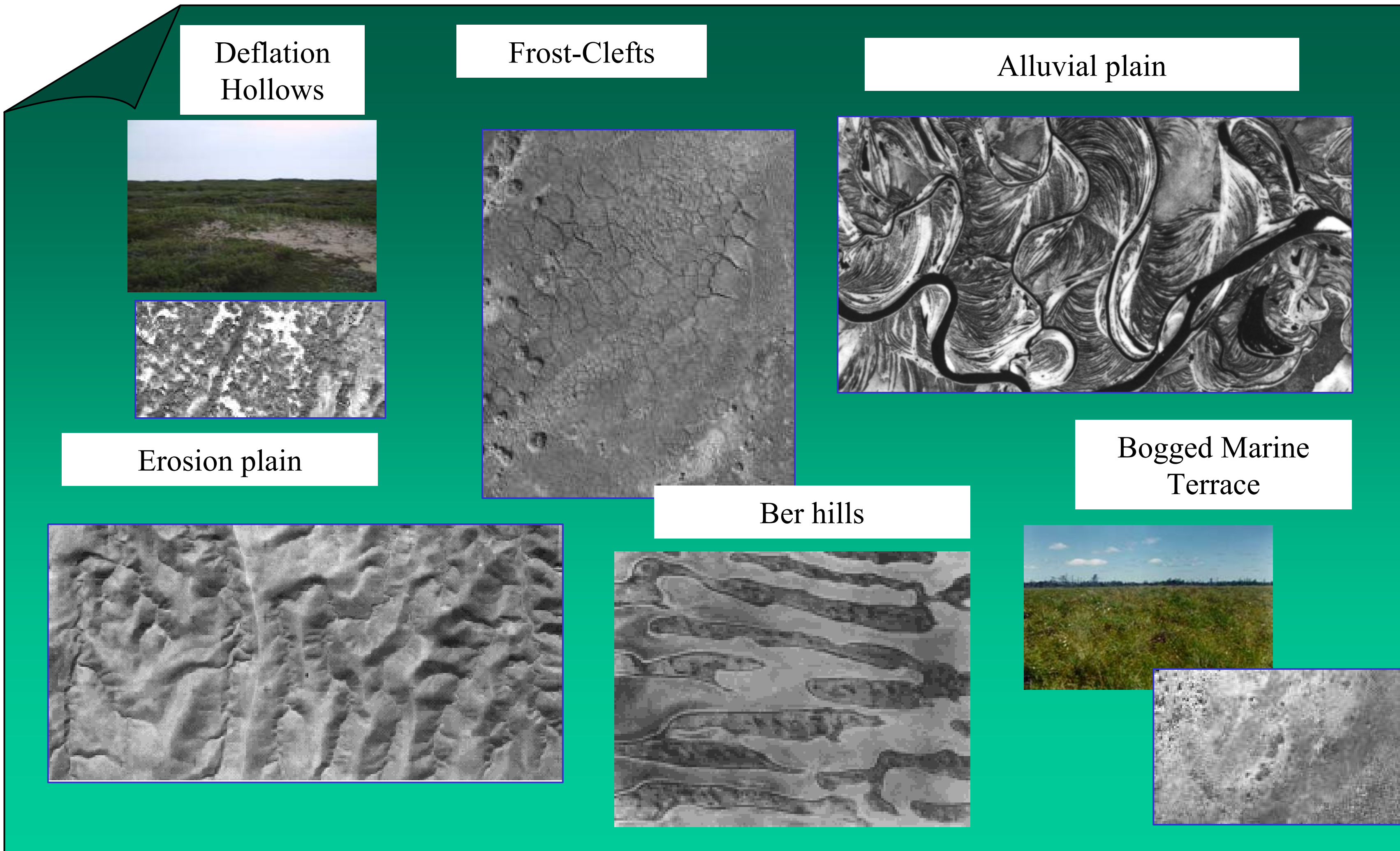
The mathematical morphology of landscape

The mathematical morphology of landscape is a scientific branch studying both numerical regularities (laws) of landscape patterns and methods of their mathematical analysis.

A great number of both scientific and practical problems require using mathematical morphology of landscape . They are:

- **Prognosis and retrospective of nature dynamics,**
- **Quantitative assessment of nature hazards and risks,**
- **Developing new interpretation techniques for remote sensing data**

Landscapes spatial pattern is a mosaic formed by natural units on the Earth surface



The mathematical model of a landscape pattern

The basic concept of mathematical morphology is a mathematical model of a landscape pattern.

Mathematical models of landscape patterns comprise a set of equations describing behavior of general numerical characteristics of a landscape pattern of a certain genesis.

The scheme of the model design includes:

- **analysis of the process mechanism and formulation of the model assumptions,**
- **mathematical analysis of the assumptions in order to obtain the main statements describing the landscape pattern behavior,**
- **empirical testing of the results with the help of remote sensing images of corresponding key areas.**

The mathematical model of a landscape pattern

Canonical mathematical models of landscape patterns are the mathematical models of morphological patterns formed by a single process in the uniform physiographic condition.

Hence, the canonical mathematical models of morphological patterns are the units that form a mathematical model of a landscape pattern for any area.

Canonical mathematical models of landsape patterns

- Landscape pattern of alluvial plains,
- Landscape pattern of plains with fluvial erosion,
- Landscape pattern of plains with karst and soil subsidence processes
- Landscape pattern of cell and ridged and ridged aeolian plains
- Landscape pattern of thermokarst lake plains
- Landscape pattern of bogged and solonchack plains
- Landscape pattern of plains with sheet erosion

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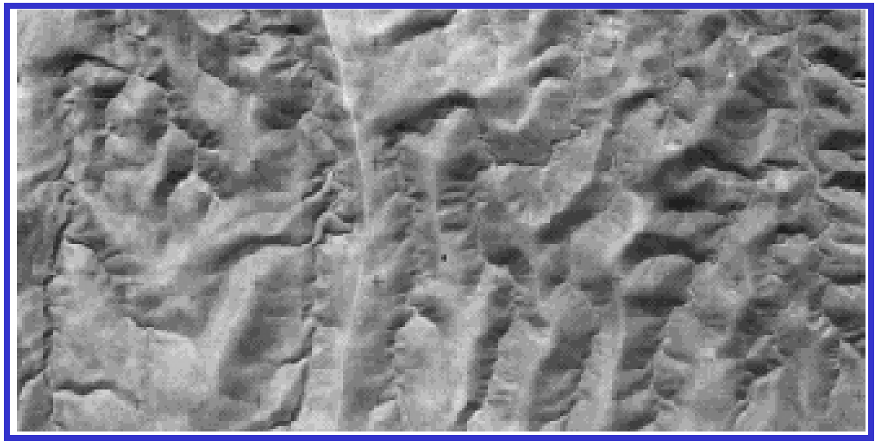
The canonical mathematical model of a landscape pattern plains with fluvial erosion



Main assumptions of the model

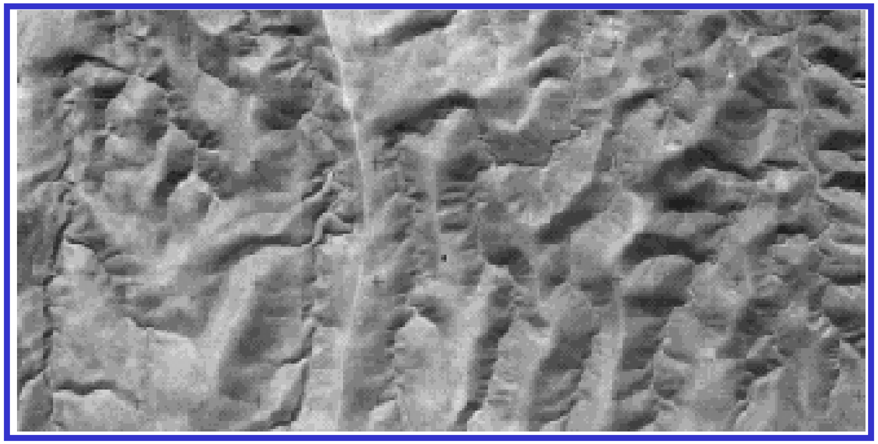
а) Two adjacent streams confluence depends neither on their order nor other steams, and its probability is directly proportional to movement down slope.

б) New source origin at different sites occurs independently of each other and existing streams and its probability is directly proportional to an area of the site.

в) New first-order streams appear in a number providing for a constant (in average) density of fluvial erosion units.

The canonical mathematical model of a landscape pattern plains with fluvial erosion

Results of the analysis



probability of different order streams (parts of the streams) down the slope

$$\frac{dp_1}{dx} = \lambda(1 - 2p_1)$$

$$\frac{dp_i}{dx} = 2\lambda(p_1 + \ldots + p_{i-1} - 1)p_i + \lambda p^2{}_{i-1}$$

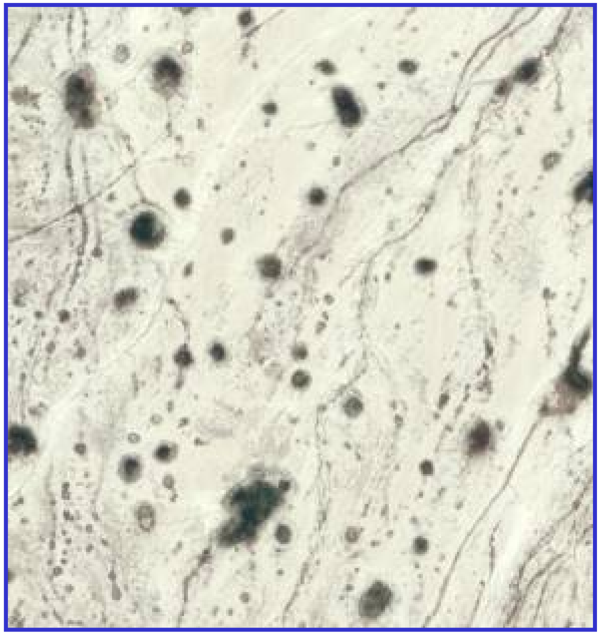
a number of streams crossing an occasional length

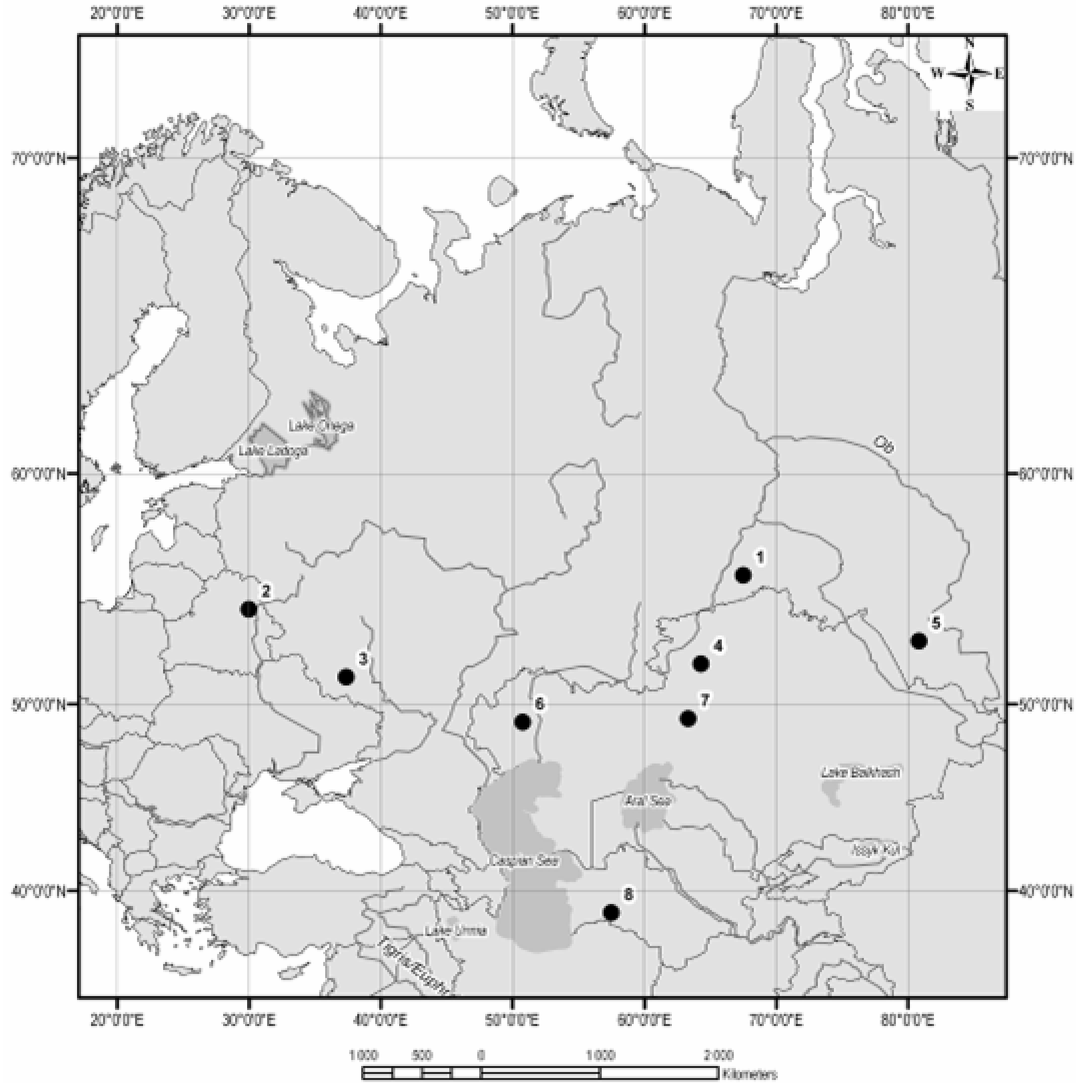
$$p_k^* = \frac{1}{k!}\left(\frac{\lambda_0 s}{\lambda}\right)^k e^{-\frac{\lambda_0 s}{\lambda}}$$

size distribution for stream basins up the slope

$$F(x) = 1 - \frac{1}{1 + \lambda x}$$

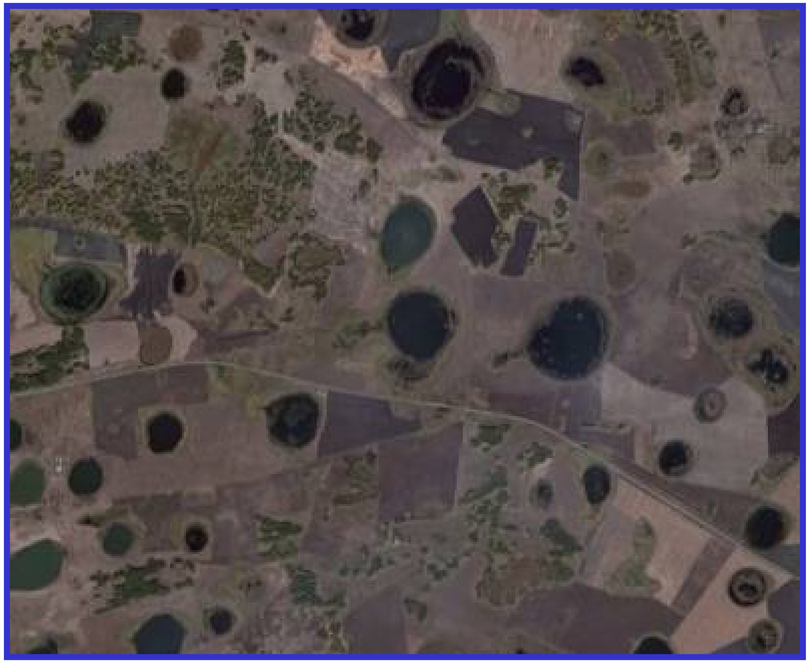
Analyzed landscape : plains developing under soil subsidence process





Map of testing areas location: 1- The south of West Siberia, 2 – Russian plain, 3 - Russian plain (Byelorussia), 4 - Turgai table land (north part), 5 - Barabinskaya steppe, 6 - Caspian lowland, 7 - Turgai table land (the southern part), 8 - Kopet-Dag piedmont plain.

Analyzed landscape : plains developing under soil subsidence process



a

b

Typical remote sensing images of plains under soil subsidence process in the south of West Siberia (a) and piedmont plain of Kopet-Dag (b).

The metrics

$$m_1 = \frac{n}{S}$$

Average spatial density of depressions (Patch Density, class level calculation)

$$m_2 = \frac{\sigma_n}{n_0}$$

Variation coefficient for spatial density of depressions

$$m_3 = \frac{\sum_{i=1}^{n} q_i}{n}$$

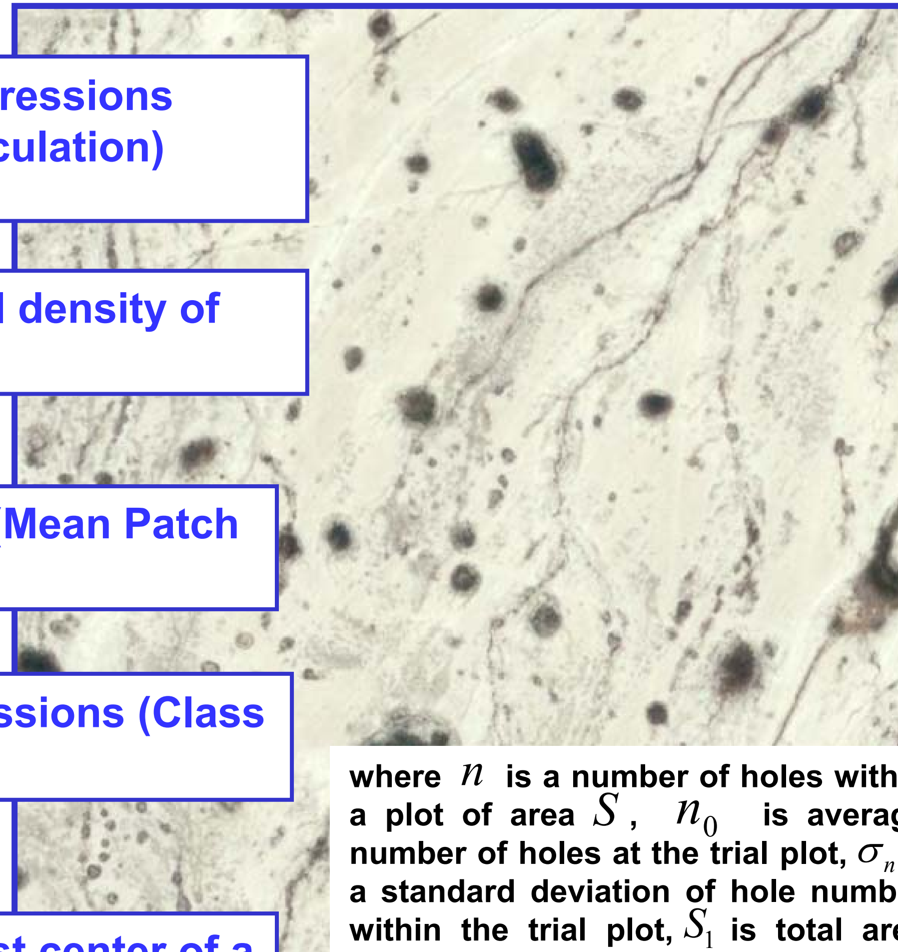
Average area of a depression (Mean Patch Size, class level calculation)

$$m_4 = \frac{S_1}{S}$$

Area share taken by the depressions (Class Area Proportion)

$$m_5 = \frac{\sum_{i=1}^{n} r_i}{n}$$

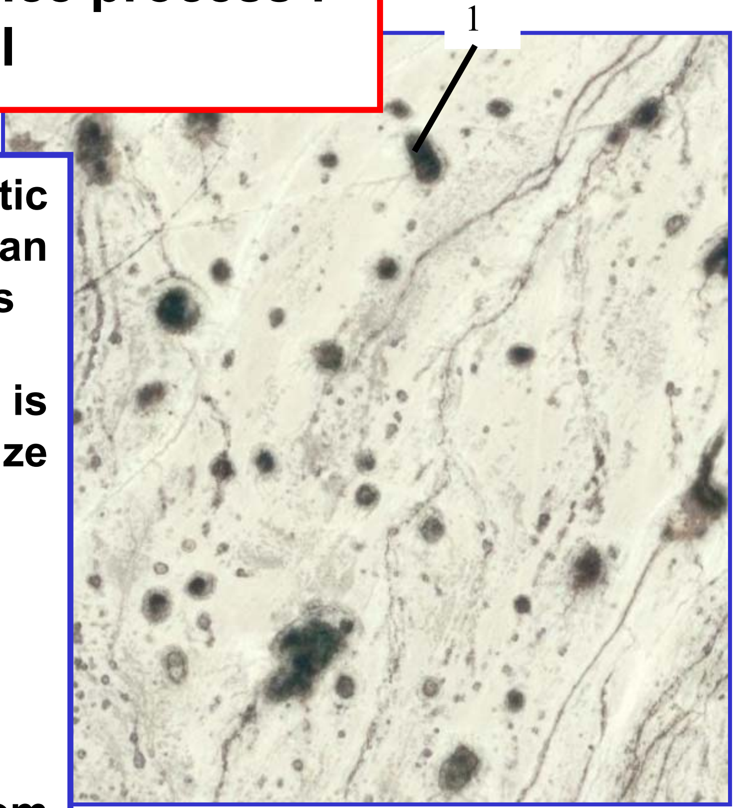
Average distance to the nearest center of a depression (Mean Euclidean Nearest Neighbor Distance)



where n is a number of holes within a plot of area S, n_0 is average number of holes at the trial plot, σ_n is a standard deviation of hole number within the trial plot, S_1 is total area taken by the holes within the plot, q_i is the area of the depression (i), r_i is a distance between the center of depression (i) to the closest one.

Is this set of metrics suitable for a joint use?
What particular metrics should we take?

Plains developing under soil subsidence process : assumptions of the model



1 – soil subsidence .

1.The occurrence of a new hole is a probabilistic process and within non-overlapping areas is an event independent of other depression occurrences

2. The probability of depression occurrence is directly proportional to the time period (Δt) and size of the area (Δs) under study.

$$p_1 = \lambda \Delta s \Delta t + o(\Delta s \Delta t)$$

$$p_k = o(\Delta s \Delta t) \quad k = 2,3...$$

3. Growth of an appeared depression is a random variable; it is independant of other depressions and the growth rate (Δx_i) is directly proportional to depression radius (x_i).

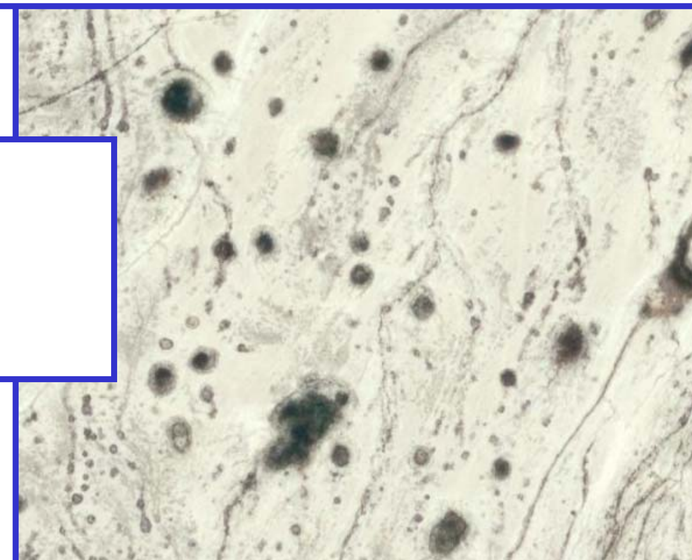
$$\Delta x_i = \xi_i x_i$$

4. A new hole cannot appear within an existing depression.

where λ is a parameter

Plains developing under soil subsidence process : analysis of the model

The distribution of a number of depressions originated for time t at a site s



$$P(k) = \frac{(\lambda st)^k}{k!} e^{-\lambda st}$$

where λ is an average number of derisions appearing within an area unit for a time unit.

Victorov A.S. (1998) Matematicheskaya morfologiya landshafta (Mathematical morphology of landscape). Tratek, Moscow, p 180

Victorov A.S. (2003) An integrated mathematical model for diffuse exogenous geological processes. Proceeding of the 9th Annual Conference of the IAMG, Portsmouth, GB.

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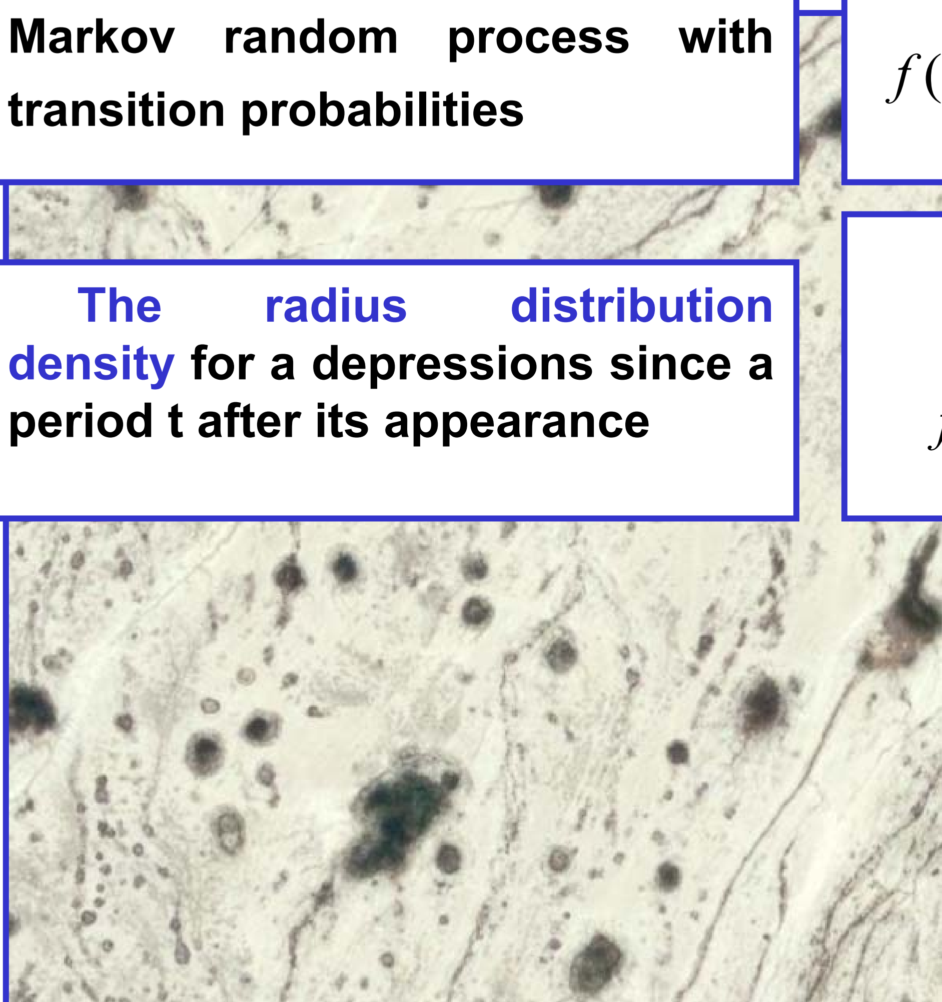


The growth of depressions radii can be considered as a Markov random process with transition probabilities

$$f(y,x,t)=\frac{1}{\sqrt{2\pi}\,\sigma_0 x\sqrt{t}}\,e^{-\frac{\left(\ln\frac{x}{y}-a_0 t\right)^2}{2\sigma_0^2 t}}$$

The radius distribution density for a depressions since a period t after its appearance

$$f_r(x,t)=\frac{1}{\sqrt{2\pi}\,\sigma_0 x\sqrt{t}}\,e^{-\frac{\left(\ln x-a_0 t\right)^2}{2\sigma_0^2 t}}$$

where

y is an initial radius,

x is a finite radius,

t is a growth period,

a_0, σ_0 are distribution parameters.

The mathematical analysis of the model assumptions enable us to draw uniquely [Victorov, 1998, 2006] the general equations of the landscape pattern model for plains under soil subsidence process. Let us demonstrate the radii distribution density of subsidence holes as an example.

Let us examine area change for a hole of soil subsidence genesis. The second assumption of the model causes proportional dependence of a generally random change of the hole radii during i section of time from the actual radii

$$\Delta u_i = \xi_i u_i \qquad (2)$$

The proportional indices are random parameters resulting from atmospheric characteristics of every year such as precipitation, storm run-off, melting water volume and others. These indices are independent from year to year but they have got the same distribution, so we can get their mathematical expectation and variance as follows:

$$M\xi_i = a_0, \quad D\xi_i = \sigma_0^2 \qquad (3)$$

By transposing the terms dealing with the hole size to the left, then summing and replacing of the left part with the integral we get

$$\int_{x_0}^{x} \frac{du}{u} = \sum_{i=1}^{t} \xi_i \qquad (4)$$

After integration we get

$$\ln x - \ln x_0 = \sum_{i=1}^{t} \xi_i \qquad (5)$$

where x is a hole size at time t, x_0 is an initial hole size, which can be taken as 1 for simplification. In this case the expression becomes simpler

$$\ln x = \sum_{i=1}^{t} \xi_i \qquad (6)$$

Taking into account the sum of a great number of independent summands on the left we get according to the central limit theorem the size logarithm as a random variable with normal distribution. The mathematical expectation of this variable and its variance grow lineally by time due to independence of the summands

$$a(t) = a_0 t, \quad \sigma^2(t) = \sigma_0^2 t \qquad (7)$$

Thus the radii distribution density of subsidence holes (and their area correspondingly) after time t since the depression appearance is

$$f_r(x,t) = \frac{1}{\sqrt{2\pi}\sigma_0 x \sqrt{t}} \exp\left[-\frac{(\ln x - a_0 t)^2}{2\sigma_0^2 t}\right], \qquad (8)$$

where a_0, σ_0 are parameters.

Interrelations among the metrics and model parameters

$$m_1 = \frac{n}{S}$$

$$m_2 = \frac{\sigma_n}{n_0}$$

$$m_3 = \frac{\sum_{i=1}^{n} q_i}{n}$$

$$m_4 = \frac{S_1}{S}$$

$$m_5 = \frac{\sum_{i=1}^{n} r_i}{n}$$

$$m_1 = \gamma$$

$$m_2 = \frac{1}{\sqrt{\gamma \; S_p}}$$

$$m_3 = \mu, \qquad \mu = \exp\left(0.5\sigma^2 + a\right)$$

$$m_4 = 1 - \exp(-\gamma \; \mu)$$

$$m_5 = \frac{1}{2\sqrt{\gamma}} + \frac{\sqrt{\gamma}}{2}\mu$$

$$\lambda$$

$$a_0$$

$$\sigma_0$$

Interrelations among the metrics: theoretical analysis

$$m_1 = \gamma$$

$$m_2 = \frac{1}{\sqrt{\gamma \, S_p}}$$

$$m_3 = \mu,$$

$$m_4 = 1 - \exp(-\gamma \, \mu)$$

$$m_5 = \frac{1}{2\sqrt{\gamma}} + \frac{\sqrt{\gamma}}{2}\mu$$

$$m_2 = \frac{1}{\sqrt{m_1 \, S_p}}$$

$$m_4 = 1 - \exp(-m_1 m_3)$$

$$m_5 = \frac{1 + m_1 m_3}{2\sqrt{m_1}}$$

The mathematical analysis of the expressions corresponding to the metrics shows hidden dependencies within the set of the metrics in question.

Interrelations among the metrics: empirical testing

Comparison between theoretical dependence of m_1 m_3 , and m_4 metrics and empirical data

$$m_4 = 1 - \exp(-m_1 m_3)$$

Region of a testing area	Metric, km-2 m_1	Metric, m_3 km2	Metric m_4	
			Empirical value	Theoretical (calculated) value
Turgai table land (the southern part)	0,111	0,820	0,106	0,087
Caspian lowland	1388,889	0,0002	0,209	0,188
Barabinskaya steppe	0,899	0,307	0,198	0,241
Caspian lowland	11,364	0,008	0,070	0,090
Kopet-Dag piedmont plain	81,439	0,001	0,053	0,073
Russian plain (Byelorussia)	148,448	0,002	0,250	0,224
The south of West Siberia	0,272	0,434	0,093	0,111
Turgai table land (the northern part)	0.364	0.354	0.053	0.129

Interrelations among the metrics: empirical testing

Comparison between theoretical dependence of m_1, m_3 and m_5 metrics and empirical data

$$m_5 = \frac{1 + m_1 m_3}{2\sqrt{m_1}}$$

m_1

Region of a testing area	Metric m_1, km^{-2}	Metric m_3, km2	Metric m_5,	
			Empirical value	Theoretical (calculated) value
Russian plain (Byelorussia)	148.448	0.002	60	51
Kopet-Dag piedmont plain	81.439	0.001	68	60
Turgai table land (the northern part)	0.364	0.355	1048	936
The south of West Siberia	0.272	0.434	1152	1072

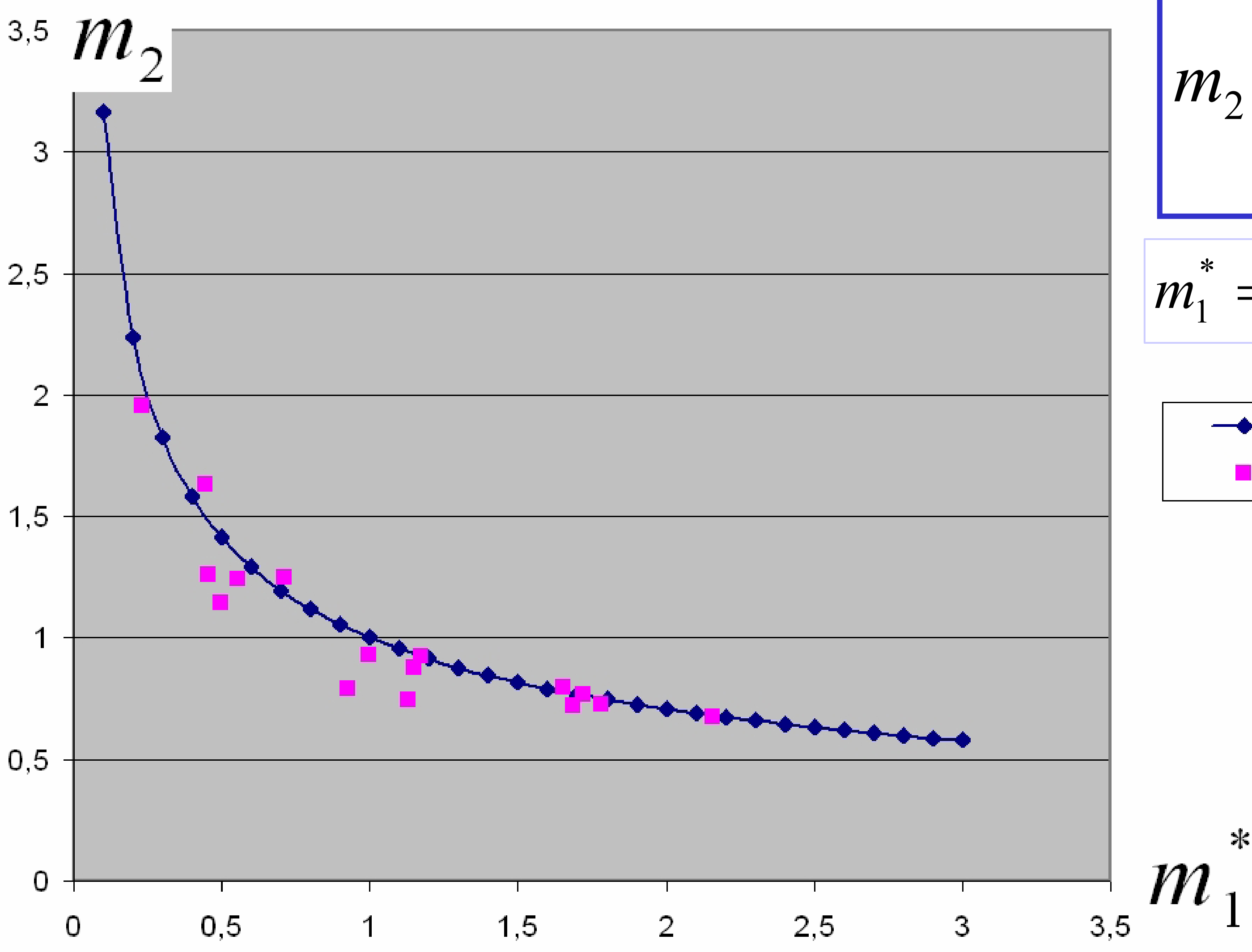
Interrelations among the metrics: empirical testing

Comparison between theoretical dependence of m_1 and m_2 metrics and empirical data

$$m_2 = \frac{1}{\sqrt{m_1 \, S_p}}$$

Region of a testing area	Metric m_1 km-2	Trial plot space, S_p km2	Metric m_2 Empirical value	Theoretical (calculated) value
Turgai table land (the southern part)	0,1113	4,0000	1,6311	1,4985
Russian plain	12,5000	0,0400	1,1439	1,4142
	12,5000	0,0800	0,9265	1,0000
Russian plain (Byelorussia)	143,9095	0,0080	0,8772	0,9325
Kopet-Dag piedmont plain	228,9060	0,0072	0,7922	0,7776
	246,8281	0,0072	0,7271	0,7488
	289,6250	0,0059	0,7622	0,7625
Caspian lowland	11,3636	0,0400	1,2573	1,4832
	11,5741	0,0800	0,7883	1,0392
	1388,888	0,0004	1,2428	1,3416
Barabinskaya steppe	0,8988	0,2601	1,9500	2,0683
	162,4793	0,0072	0,9198	0,9230
	18,6894	0,1156	0,6726	0,6803
	24,6812	0,0289	1,2439	1,1840
Turgai table land (the northern part)	0,3641	3,1038	0,7405	0,9407
The south of West Siberia	0,2719	6,2151	0,7171	0,7692

Interrelations among the metrics: empirical testing



$$m_2 = \frac{1}{\sqrt{m_1^*}}$$

$$m_1^* = m_1 \; S_p$$

$$m_1 = \frac{n}{S}$$

Average spatial density of depressions (Patch Density, class level calculation)

$$m_2 = \frac{\sigma_n}{n_0}$$

Variation coefficient for spatial density of depressions

$$m_3 = \frac{\sum_{i=1}^{n} q_i}{n}$$

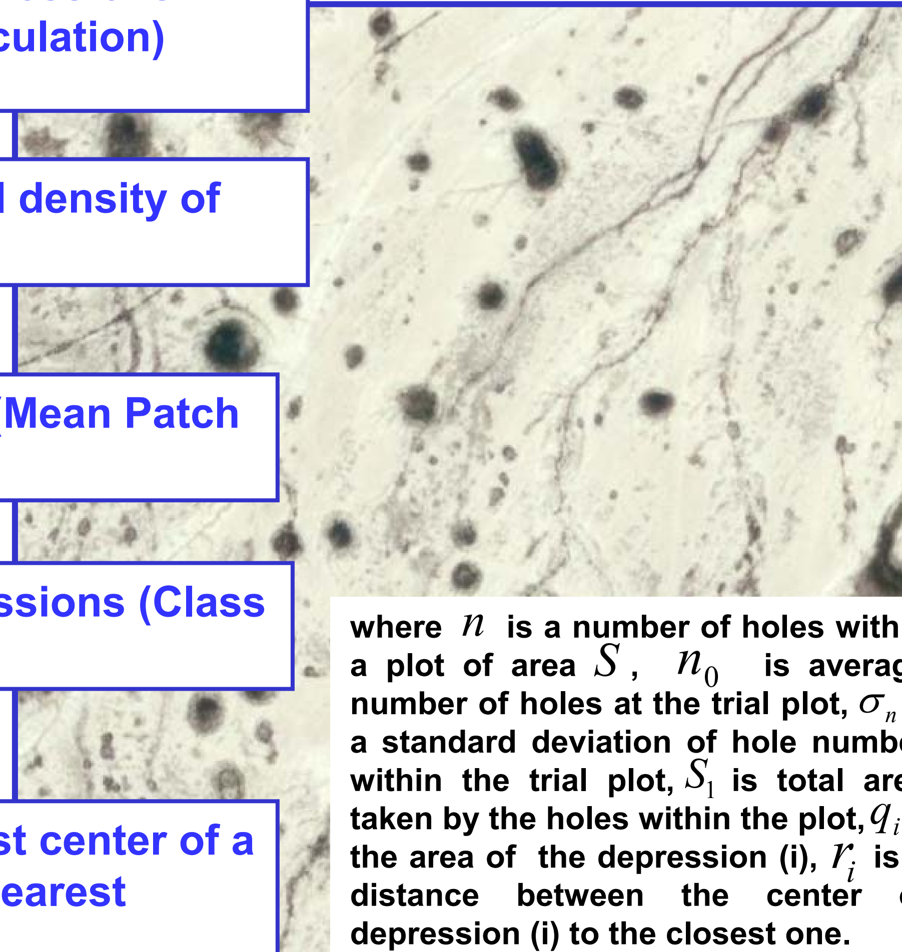
Average area of a depression (Mean Patch Size, class level calculation)

$$m_4 = \frac{S_1}{S}$$

Area share taken by the depressions (Class Area Proportion)

$$m_5 = \frac{\sum_{i=1}^{n} r_i}{n}$$

Average distance to the nearest center of a depression (Mean Euclidean Nearest Neighbor Distance)



where n is a number of holes within a plot of area S, n_0 is average number of holes at the trial plot, σ_n is a standard deviation of hole number within the trial plot, S_1 is total area taken by the holes within the plot, q_i is the area of the depression (i), r_i is a distance between the center of depression (i) to the closest one.

It means that the given set of metrics is not optimal because the second, forth and fifth metrics are non-informative as far they are related in unobvious way with the first and third metrics and are under their control.

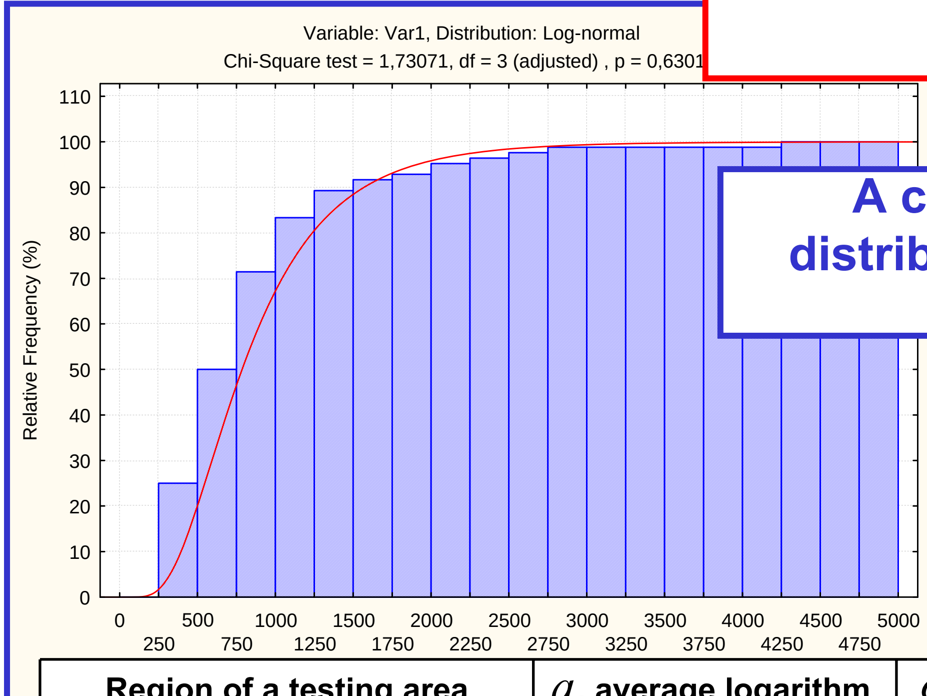
Model Invariants

$$P(k) = \frac{\gamma^k}{k!} e^{-\gamma}$$

A comparison of actual and theoretical distribution for depressions density with the Poisson Law

Region of a testing area	γ	N	χ^2	$\chi^2_{0.99}$
Caspian lowland	0.548	62	0.056	6.635
Caspian lowland	0.912	80	0.262	6.635
Kopet-Dag piedmont plain	1,654	130	0.494	13,277
	3.308	65	2.736	15,086
	1.720	100	1.858	11.341
North Africa (Algeria)	0.622	98	2.924	6.635
Barabinskaya steppe	1.174	69	1.056	9,210
-“-	0.520	324	1.676	6.635
-“-	0.713	143	0.158	6.635
-“-	2.123	81	1.766	13,277
Russian plain	1.000	54	1.191	6.635
Russian plain (Byelorussia)	1.150	100	9.250	9,210
Orsha-Mogilev plateau (Byelorussia)	1.044	206	10.671	9,210
The south of West Siberia	1.690	100	5.121	11.341

Model Invariants



A comparison of actual and theoretical distribution for depressions density with the lognormal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - a)^2}{2\sigma^2}}$$

Region of a testing area	a, average logarithm of depression	σ, standard deviation for logarithm of depressions	N	χ^2	$\chi^2_{0.99}$
The south of West Siberia	12.685	0.734	78	1.329	9.210
Russian plain (Byelorussia)	3,980	0,409	61	3.889	9.210
	1,294	0,522	59	2.469	9.210
	0,153	0,635	52	4.163	9.210
Kopet-Dag piedmont plain	0.668	0.290	84	1.731	11.341
Turgai table land (the northern part)	12.077	1.325	44	4.739	6.635
Caspian lowland	4.537	0.955	34	0.763	9,210

The optimizing selection of landscape metrics based on the mathematical models of landscape patterns remains true for plains under soil subsidence process (area of the same genetic type) in wide spectrum of nature conditions.

Conclusions

- **The problems of landscape metrics selection can be solved basing on the new branch of landscape science, the mathematical morphology of landscape, using the mathematical models of landscape patterns.**

- **The optimizing selection of landscape metrics based on the mathematical models of landscape patterns remains true for any area of the same genetic type in wide spectrum of nature conditions.**

- **The carried research demonstrates high availability of mathematical analysis of the landscape metrics behavior basing on the mathematical models of the mathematical morphology of landscape.**